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Supersymmetric extension of the scalar Born–Infeld equation

A J Hariton[1]

Département de mathématiques et statistique, Université de Montréal, CP 6128, Succ. Centre-ville, Montréal, QC H3C 3J7, Canada

E-mail: hariton@lns.mit.edu

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Abstract

In this paper, a supersymmetric extension of the scalar Born–Infeld equation is constructed through a superspace formalism which involves the addition of one independent fermionic Grassmann variable to the existing bosonic spacetime coordinates. The bosonic scalar field is replaced by a fermionic superfield which is composed of two component fields, one bosonic and one fermionic. The resulting equation is invariant under a space supersymmetric transformation and, in its most general form, involves four arbitrary parameters. For a certain specific case, the Lie superalgebra of symmetries was identified, and the one-dimensional subalgebras were systematically classified into splitting and non-splitting conjugate classes. A number of group-invariant solutions were obtained, including polynomial solutions, solutions by radicals and solitary waves (including bumps, kinks and doubly periodic solutions).

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1. Introduction

Over the last three decades, since the discovery of supersymmetry by Golfand and Likhtman [1], there has been a great deal of interest in the subject of supersymmetric theories in physics. This has led to the appearance of new types of differential equations involving Grassmann variables. At the same time, certain kinds of symmetries of these equations have been identified which link the bosonic and fermionic Grassmann variables in a nontrivial way. These elements have made possible the formulation of different types of supersymmetric theories to describe a number of phenomena in both classical and quantum physics [2–12].

[1] Present address: Massachusetts Institute of Technology, Center for Theoretical Physics, NE25-4113, 77 Massachusetts Avenue, Cambridge, MA 02139, USA.

1.1. Scalar Born–Infeld equation

In 1934, Born and Infeld proposed a nonlinear modification of Maxwell's electrodynamics theory [13]. More generally, equations derived from an action principle involving a square root similar to that of the Born–Infeld Lagrangian are referred to as Born–Infeld type equations in the literature on the subject [14–18]. The focus of this paper will be on the following nonlinear differential equation involving the scalar field $\phi(x,t)$:

$$(1+(\phi_x)^2)\phi_{tt} = 2\phi_x\phi_t\phi_{xt} + (1-(\phi_t)^2)\phi_{xx}, \tag{1}$$

which will be referred to throughout as the scalar Born–Infeld (SBI) equation. This equation, which is derived from the variational principle

$$\delta \int\!\!\int (1-(\phi_t)^2+(\phi_x)^2)^{1/2}\,\mathrm{d}x\,\mathrm{d}t = 0, \tag{2}$$

was used by Barbashov and Chernikov to formulate and solve the problem of the interaction of two plane waves [15]. Equation (1) itself was introduced as an analogy to the equation of minimal surfaces in Euclidean space

$$(1+(\phi_x)^2)\phi_{yy} - 2\phi_x\phi_y\phi_{xy} + (1+(\phi_y)^2)\phi_{xx}, \tag{3}$$

by setting $y = \mathrm{i}t$. The integral in (2) expresses the area of the surface $z = \phi(x,t)$ in the pseudo-Euclidean space with the metric $\mathrm{d}s = \mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}z^2$, and accordingly, equation (1) describes the extremal surfaces in pseudo-Euclidean space [14]. The scattering problem was solved exactly and it was found that the shape and direction of the plane waves do not change after scattering, only their arguments [15].

It was later determined that both the scalar Born–Infeld theory and the Chaplygin gas nonrelativistic fluid devolve from the Nambu–Goto action for a d-brane, when specific parametrizations are made [9, 19, 20]. More specifically, the Chaplygin gas is derived through the light-cone parametrization whereas the Born–Infeld model is obtained from the Cartesian parametrization. This connection between the three models (Chaplygin, Born–Infeld and Nambu–Goto) has been used extensively in the context of symmetry analysis, conservation laws and the construction of invariant and partially invariant solutions [9, 21, 22]. Supersymmetric extensions of the Chaplygin gas model in one and two spatial dimensions have been formulated recently by Jackiw, Bergner and Polychronakos [9–11]. The question arises as to whether it is possible to find a supersymmetric generalization of the associated scalar Born–Infeld equation. The construction of such a model is the main objective of this paper.

1.2. Objectives and organization

The primary purpose of this paper is to construct a supersymmetric extension of the scalar Born–Infeld (SBI) equation by means of a superspace and superfield formalism. Such a superfield generalization is not always unique however. Indeed, due to the fact that each term of the SBI equation may be the fermionic part of more than one combination of derivatives of the superfield, the most general possible supersymmetric extension must include linear combinations of all possible terms. For a certain choice of linear combination, we wish to determine the Lie superalgebra L_S of symmetries of the resulting supersymmetric generalization and perform a comprehensive classification of the one-dimensional subalgebras of L_S. This classification allows us to construct group-invariant solutions of the supersymmetric SBI system which we have constructed.

This paper is organized as follows. Section 2 describes in detail the procedure utilized to construct the most general supersymmetric extension of the SBI equation. The Lie symmetries of the resulting supersymmetric model are described in section 3 for both the general case and a specific case which we describe in greater detail. For the latter, we also perform a complete classification of the splitting and non-splitting one-dimensional subalgebras of the symmetry Lie superalgebra. In section 4, this classification is used together with the method of symmetry reduction in order to obtain certain classes of group-invariant solutions of our equation. Finally, section 5 contains observations and a discussion of future perspectives.

2. Supersymmetric Born–Infeld equation

We now proceed to construct an explicit Grassmann-valued extension of the SBI equation. We extend the space of independent variables $\{(x, t)\}$ to a superspace $\{(x, t, \theta)\}$, where θ is an anticommuting Grassmann variable. The bosonic field $\phi(x, t)$ is generalized to a fermionic superfield $\Phi(x, t, \theta)$, which is defined as

$$\Phi(x, t, \theta) = \psi(x, t) + \theta\phi(x, t), \tag{4}$$

where $\psi(x, t)$ is a new fermionic field. The new system is constructed in such a way that it is invariant under the supersymmetry transformation

$$x \to x - \underline{\eta}\theta, \qquad \theta \to \theta + \underline{\eta}, \tag{5}$$

which in turn is generated by the infinitesimal supersymmetry operator

$$Q = \partial_\theta - \theta\partial_x. \tag{6}$$

In addition to the superfield Φ described in (4), we introduce the covariant derivative

$$D = \partial_\theta + \theta\partial_x. \tag{7}$$

The most general form of the supersymmetric extension of (1) is given by the expression

$$\begin{aligned} \Phi_{tt} + a(D^3\Phi)^2\Phi_{tt} + (1-a)(D\Phi)_{tt}(D^2\Phi)(D^3\Phi) &= 2b(D\Phi)_t(D^3\Phi)(D^2\Phi)_t \\ &+ 2c(D\Phi)_t(D^2\Phi)(D^3\Phi)_t + 2(1-b-c)\Phi_t(D^3\Phi)(D^3\Phi)_t \\ &+ D^4\Phi - d((D\Phi)_t)^2(D^4\Phi + (d-1)\Phi_t(D\Phi)_t(D^5\Phi). \end{aligned} \tag{8}$$

When decomposed in terms of the component fields of Φ, equation (8) is equivalent to the following system of two partial differential equations for the two unknown fields ϕ and ψ expressed as functions of x and t:

$$\begin{aligned} (1+(\phi_x)^2)\phi_{tt} &= 2\phi_x\phi_t\phi_{xt} + (1-(\phi_t)^2)\phi_{xx} - 2a\phi_x\psi_{xx}\psi_{tt} + (1-a)\phi_{tt}\psi_x\psi_{xx} \\ &+ (1-a)\phi_x\psi_x\psi_{xtt} + 2(b+d)\phi_t\psi_{xx}\psi_{xt} - 2c\phi_t\psi_x\psi_{xxt} - 2c\phi_{xt}\psi_x\psi_{xt} \\ &+ 2(1-b-c)\phi_x\psi_{xxt}\psi_t + 2(1-b-c)\phi_{xt}\psi_{xx}\psi_t + (1-d)\phi_{xx}\psi_t\psi_{xt} \\ &+ (1-d)\phi_t\psi_t\psi_{xxx} \end{aligned}$$

and

$$\begin{aligned} \psi_{tt} + a(\phi_x)^2\psi_{tt} + (1-a)\phi_x\phi_{tt}\psi_x &= 2b\phi_x\phi_t\psi_{xt} + 2c\phi_t\phi_{xt}\psi_x + 2(1-b-c)\phi_x\phi_{xt}\psi_t \\ &+ \psi_{xx} - d(\phi_t)^2\psi_{xx} + (d-1)\phi_t\phi_{xx}\psi_t. \end{aligned} \tag{9}$$

This pair of equations represents the supersymmetric scalar Born–Infeld (SSBI) system and we will refer to it as such.

3. Structure of the symmetry Lie superalgebra

The system composed of the two SSBI equations (9) possesses a number of infinitesimal symmetries which are present in general, for all possible values of the real parameters a, b, c and d. These include the dilation in dependent and independent variables

$$M = x\partial_x + t\partial_t + \phi\partial_\phi + \tfrac{3}{2}\psi\partial_\psi \tag{10}$$

and the translations in time t and space x as well as in the bosonic and fermionic fields ϕ and ψ, respectively,

$$P_0 = \partial_t, \qquad P_1 = \partial_x, \qquad Z = \partial_\theta, \qquad Y = \partial_\psi. \tag{11}$$

The dilation and translations listed in (10) and (11) represent an extension of the symmetry generators of the standard SBI equation (1). For specific values of the coefficients a, b, c and d, certain additional symmetry generators may be present and consequently we would get a higher dimensional Lie superalgebra. We examine such an example below.

It should be noted that the supersymmetry operator Q given in equation (6), which connects the independent variables x and θ in the (x, t, θ) superspace can also be represented as a generalized symmetry in (x, t, ϕ, ψ) coordinate space. Indeed, if $\underline{\eta}$ is a constant fermionic parameter, the superfield $\Phi = \psi + \theta\phi$ is transformed under the action of $\underline{\eta}Q$ to

$$\Phi \to \Phi + \underline{\eta}Q\Phi = (\psi + \underline{\eta}\phi) + \theta(\phi - \underline{\eta}\psi_x), \tag{12}$$

so that Q may be represented by the operator

$$\hat{Q} = -\psi_x\partial_\phi + \phi\partial_\psi. \tag{13}$$

Similarly, the covariant derivative D given in equation (7) can be represented by the generalized symmetry

$$\hat{D} = \psi_x\partial_\phi + \phi\partial_\psi. \tag{14}$$

3.1. The case where $a = 1, b = 1, c = 0$ and $d = 1$

For the rest of this paper, we concentrate on the case where the parameters in equations (9) are chosen to be $a = 1, b = 1, c = 0$ and $d = 1$, for which certain additional symmetry generators are present. For this case, the supersymmetric extension (8) of the SBI equation in terms of the superfield Φ is given by the expression

$$(1 - ((D\Phi)_t)^2)(D^4\Phi) + 2(D\Phi)_t(D^3\Phi)(D^2\Phi)_t - (1 + (D^3\Phi)^2)\Phi_{tt} = 0, \tag{15}$$

and the equivalent component equations (9) are now

$$(1 - (\phi_t)^2)\phi_{xx} + 2\phi_x\phi_t\phi_{xt} - (1 + (\phi_x)^2)\phi_{tt} + \frac{4\phi_t}{(1 + (\phi_x)^2)}\psi_{xx}\psi_{xt} = 0$$

and (16)

$$(1 - (\phi_t)^2)\psi_{xx} + 2\phi_x\phi_t\psi_{xt} - (1 + (\phi_x)^2)\psi_{tt} = 0.$$

Here, we have eliminated the ψ_{tt} term in the first equation of (16) by substituting the isolated ψ_{tt} term from the second equation.

The Lie superalgebra L_S of the system composed of the two equations (16) is spanned by the seven independent vector fields:

$$\begin{aligned} &P_0 = \partial_t, \qquad P_1 = \partial_x, \qquad Z = \partial_\theta, \qquad M = x\partial_x + t\partial_t + \phi\partial_\phi + \tfrac{3}{2}\psi\partial_\psi, \\ &Y = \partial_\psi, \qquad Q_0 = t\partial_\psi, \qquad Q_1 = x\partial_\psi. \end{aligned} \tag{17}$$

Table 1. Supercommutation table for the Lie superalgebra L_S spanned by the vector fields (17).

$A\backslash B$	$\mathbf{P}_0$	$\mathbf{P}_1$	$\mathbf{Z}$	$\mathbf{M}$	$\mathbf{Y}$	$\mathbf{Q}_0$	$\mathbf{Q}_1$
$\mathbf{P}_0$	0	0	0	P_0	0	Y	0
$\mathbf{P}_1$	0	0	0	P_1	0	0	Y
$\mathbf{Z}$	0	0	0	Z	0	0	0
$\mathbf{M}$	$-P_0$	$-P_1$	$-Z$	0	$-\frac{3}{2}Y$	$-\frac{1}{2}Q_0$	$-\frac{1}{2}Q_1$
$\mathbf{Y}$	0	0	0	$\frac{3}{2}Y$	0	0	0
$\mathbf{Q}_0$	$-Y$	0	0	$\frac{1}{2}Q_0$	0	0	0
$\mathbf{Q}_1$	0	$-Y$	0	$\frac{1}{2}Q_1$	0	0	0

The new fermionic vector fields Q_0 and Q_1 correspond to fermionic transformations linking the independent bosonic variables x and t to the fermionic field ψ:

$$Q_0 : \psi \to \psi + \underline{\eta} t, \qquad Q_1 : \psi \to \psi + \underline{\eta} x, \tag{18}$$

where in either case $\underline{\eta}$ is a fermionic constant parameter. The supercommutation relations of the superalgebra L_S are given in table 1. Here, for each pair of generators of A and B, we calculate either the commutator $[A, B] = AB - BA$ if either A or B is bosonic, or the anticommutator $\{A, B\} = AB + BA$ if both A and B are fermionic.

3.2. Classification of the one-dimensional subalgebras

We now proceed to classify the one-dimensional subalgebras of the Lie superalgebra L_S, using the procedures described in [23]. That is, we construct a list of representatives of the conjugacy classes of one-dimensional subalgebras of L_S in such a way that each one-dimensional subalgebra of L_S is conjugate to one and only one element of the list. We begin by decomposing the structure of L_S into the following semi-direct sum:

$$L_S = \{\{M\} \sqsupset\!\!\!+ \{P_0, P_1, Z\}\} \quad \sqsupset\!\!\!+ \quad \{Y, Q_0, Q_1\}. \tag{19}$$

Next, we apply the classification method for semi-direct sums and obtain the following results. The one-dimensional splitting subalgebras of L_S are

$$\begin{aligned} &L_1 = \{M\}, \qquad L_2 = \{P_0\}, \qquad L_3 = \{P_1\}, \qquad L_{4,m} = \{P_1 + mP_0, m \neq 0\}, \\ &L_5 = \{Z\}, \qquad L_{6,m} = \{Z + mP_0, m \neq 0\}, \qquad L_{7,m} = \{Z + mP_1, m \neq 0\}, \\ &L_{8,m,n} = \{Z + mP_0 + nP_1, m, n \neq 0\}, \qquad L_9 = \{Y\}, \qquad L_{10} = \{Q_0\}, \\ &L_{11} = \{Q_1\}, \qquad L_{12,k} = \{Q_1 + kQ_0, k \neq 0\}, \end{aligned} \tag{20}$$

and the one-dimensional non-splitting subalgebras of L_S are

$$\begin{aligned} &\mathcal{L}_{(2;\underline{u},\underline{\eta})} = \{P_0 + \underline{u}Q_0 + \underline{\eta}Q_1\}, \qquad \mathcal{L}_{(3;\underline{u},\underline{\eta})} = \{P_1 + \underline{u}Q_0 + \underline{\eta}Q_1\} \\ &\mathcal{L}_{(4,m;\underline{u},\underline{\eta})} = \{P_1 + mP_0 + \underline{u}Q_0 + \underline{\eta}Q_1, m \neq 0\}, \qquad \mathcal{L}_{(5;\underline{u},\underline{\eta})} = \{Z + \underline{u}Q_0 + \underline{\eta}Q_1\}, \\ &\mathcal{L}_{(6,m;\underline{u},\underline{\eta})} = \{Z + mP_0 + \underline{u}Q_0 + \underline{\eta}Q_1, m \neq 0\}, \\ &\mathcal{L}_{(7,m;\underline{u},\underline{\eta})} = \{Z + mP_1 + \underline{u}Q_0 + \underline{\eta}Q_1, m \neq 0\}, \\ &\mathcal{L}_{(8,m,n;\underline{u},\underline{\eta})} = \{Z + mP_0 + nP_1 + \underline{u}Q_0 + \underline{\eta}Q_1, m, n \neq 0\}, \end{aligned} \tag{21}$$

where $\underline{u}$ and $\underline{\eta}$ are fermionic constants. It should be noted that each subalgebra $\mathcal{L}_{(i;\underline{u},\underline{\eta})}$ is an element of the same conjugacy class as the subalgebra $\mathcal{L}_{(i,K\underline{u},K\underline{\eta})}$, where K is a positive constant.

Table 2. Invariants of the one-dimensional subalgebras of L_S.

Subalgebra	Invariants	Relations and change of variable
$L_1 = \{D\}$	$\xi = \frac{x}{t}, \frac{\phi}{t}, t^{-3/2}\psi$	$\phi = tF(\xi), \psi = t^{3/2}\Lambda(\xi)$
$L_2 = \{P_0\}$	x, ϕ, ψ	$\phi = \phi(x), \psi = \psi(x)$
$L_3 = \{P_1\}$	t, ϕ, ψ	$\phi = \phi(t), \psi = \psi(t)$
$L_{4,m} = \{P_1 + mP_0\}$	$\xi = t - mx, \phi, \psi$	$\phi = \phi(\xi), \psi = \psi(\xi)$
$L_5 = \{Z\}$	x, t, ψ	N/A
$L_{6,m} = \{Z + mP_0\}$	$x, t - m\phi, \psi$	$\phi = \frac{1}{m}(F(x) + t), \psi = \psi(x)$
$L_{7,m} = \{Z + mP_1\}$	$t, x - m\phi, \psi$	$\phi = \frac{1}{m}(F(t) + x), \psi = \psi(t)$
$L_{8,m,n} = \{Z + mP_0 + nP_1\}$	$\xi = x - \frac{n}{m}t, \phi - \frac{t}{m}, \psi$	$\phi = F(\xi) + \frac{t}{m}, \psi = \psi(\xi)$
$L_9 = \{Y\}$	x, t, ϕ	N/A
$L_{10} = \{Q_0\}$	x, t, ϕ	N/A
$L_{11} = \{Q_1\}$	x, t, ϕ	N/A
$L_{12,k} = \{Q_1 + kQ_0\}$	x, t, ϕ	N/A
$\mathcal{L}_{(2;\underline{u},\underline{\eta})} = \{P_0 + \underline{u}Q_0 + \underline{\eta}Q_1\}$	$x, \phi, \psi - \frac{1}{2}\underline{u}t^2 - \underline{\eta}xt$	$\phi = \phi(x), \psi = \Lambda(x) + \frac{1}{2}\underline{u}t^2 + \underline{\eta}xt$
$\mathcal{L}_{(3;\underline{u},\underline{\eta})} = \{P_1 + \underline{u}Q_0 + \underline{\eta}Q_1\}$	$t, \phi, \psi - \underline{u}xt - \frac{1}{2}\underline{\eta}x^2$	$\phi = \phi(t), \psi = \Lambda(t) + \underline{u}xt + \frac{1}{2}\underline{\eta}x^2$
$\mathcal{L}_{(4,m;\underline{u},\underline{\eta})} = \{P_1 + mP_0$ $+ \underline{u}Q_0 + \underline{\eta}Q_1\}$	$\xi = t - mx, \phi$ $\psi + \frac{1}{2}\underline{u}mx^2 - \frac{1}{2}\underline{\eta}x^2 - \underline{u}xt$	$\phi = \phi(\xi),$ $\psi = \Lambda(\xi) - \frac{1}{2}\underline{u}mx^2 + \frac{1}{2}\underline{\eta}x^2 + \underline{u}xt$
$\mathcal{L}_{(5;\underline{u},\underline{\eta})} = \{Z + \underline{u}Q_0 + \underline{\eta}Q_1\}$	$x, t, \psi - (\underline{u}t + \underline{\eta}x)\phi$	N/A
$\mathcal{L}_{(6,m;\underline{u},\underline{\eta})} = \{Z + mP_0$ $+ \underline{u}Q_0 + \underline{\eta}Q_1\}$	$x, m\phi - t,$ $\psi - \frac{1}{2m}\underline{u}t^2 - \frac{1}{m}\underline{\eta}xt$	$\phi = \frac{1}{m}(F(x) + t),$ $\psi = \Lambda(x) + \frac{1}{2m}\underline{u}t^2 + \frac{1}{m}\underline{\eta}xt$
$\mathcal{L}_{(7,m;\underline{u},\underline{\eta})} = \{Z + mP_1$ $+ \underline{u}Q_0 + \underline{\eta}Q_1\}$	$t, m\phi - x,$ $\psi - \frac{1}{m}\underline{u}xt - \frac{1}{2m}\underline{\eta}x^2$	$\phi = \frac{1}{m}(F(t) + x),$ $\psi = \Lambda(t) + \frac{1}{m}\underline{u}xt + \frac{1}{2m}\underline{\eta}x^2$
$\mathcal{L}_{(8,m,n;\underline{u},\underline{\eta})} = \{Z + mP_0 + nP_1$ $+ \underline{u}Q_0 + \underline{\eta}Q_1\}$	$\xi = x - \frac{n}{m}t, m\phi - t,$ $\psi - \frac{1}{2m}\underline{u}t^2 - \frac{1}{m}\underline{\eta}xt + \frac{b}{2m^2}\underline{\eta}t^2$	$\phi = \frac{1}{m}(F(\xi) + t),$ $\psi = \Lambda(\xi) + \frac{1}{2m}\underline{u}t^2 - \frac{1}{m}\underline{\eta}xt + \frac{b}{2m^2}\underline{\eta}t^2$

The usefulness of the classification is demonstrated in the fact that it allows us to find all corresponding reductions of the SSBI equations (16) under the classified non-equivalent one-dimensional subalgebras of L_S.

4. Group-invariant solutions

In this section, we use the classical method of symmetry reduction to determine the invariants and reduced differential equations corresponding to each subalgebra listed in section 3. Where it is possible, we also determine explicit solutions of the SSBI equations. Passing systematically through the one-dimensional subalgebras of L_S, we obtain for each subalgebra a symmetry variable ξ involving the independent variables x and t. We also express the fields ϕ and ψ in terms of the dependent invariants $F(\xi)$ and $\Lambda(\xi)$, respectively. Substituting each of these into equations (16), we reduce them to a system of reduced ordinary differential equations. It should be noted that this procedure cannot be used for subalgebras whose symmetry generators do not involve derivatives with respect to independent variables. This is also true for the generalized symmetries (13) and (14). The results are described in tables 2 and 3.

For the subalgebra L_1, the reduced equations are difficult to solve in general. However, for the specific case where $F(\xi) = \pm \mathrm{i}\xi$, we obtain the solution

$$\phi(x) = \pm \mathrm{i}x, \qquad \psi(x, t) = \eta_1 x t^{1/2} + \eta_2 t^{3/2}, \tag{22}$$

where η_1, η_2 are fermionic constant parameters.

Table 3. Reduced equations obtained from the one-dimensional subalgebras of L_S. Splitting subalgebras are denoted by L_α and non-splitting subalgebras by $\mathcal{L}_\alpha$.

Subalgebra	Reduced equation(s)
$L_1 = \{D\}$	$(F^2+\xi^2-1)(1+(F_\xi)^2)F_{\xi\xi} = 2(\xi F_\xi - F)\Lambda_\xi\Lambda_{\xi\xi}$, $(F^2+\xi^2-1)\Lambda_{\xi\xi} - (FF_\xi+\xi)\Lambda_\xi + \frac{3}{4}(1+(F_\xi)^2)\Lambda = 0$
$L_2 = \{P_0\}$	$\phi_{xx} = 0,\ \psi_{xx} = 0$
$L_3 = \{P_1\}$	$\phi_{tt} = 0,\ \psi_{tt} = 0$
$L_{4,m} = \{P_1 + mP_0\}$	$(1-m^2)\phi_{\xi\xi} = 0,\ (1-m^2)\psi_{\xi\xi} = 0$
$L_{6,m} = \{Z + mP_0\}$	$(m^2-1)F_{xx} = 0,\ (m^2-1)\psi_{xx} = 0$
$L_{7,m} = \{Z + mP_1\}$	$(m^2+1)F_{tt} = 0,\ (m^2+1)\psi_{tt} = 0$
$L_{8,m,n} = \{Z + mP_0 + nP_1\}$	$(1-m^2+n^2)F_{\xi\xi} = 0,\ (1-m^2+n^2)\psi_{\xi\xi} = 0$
$\mathcal{L}_{(2;\underline{u},\underline{\eta})} = \{P_0 + \underline{u}Q_0 + \underline{\eta}Q_1\}$	$\phi_{xx} = 0,\ \Lambda_{xx} = (1+(\phi_x)^2)\underline{u}$
$\mathcal{L}_{(3;\underline{u},\underline{\eta})} = \{P_1 + \underline{u}Q_0 + \underline{\eta}Q_1\}$	$\phi_{tt} = 4\phi_t\underline{\eta}\underline{u},\ \Lambda_{tt} = (1-(\phi_t)^2)\underline{\eta}$
$\mathcal{L}_{(4,m;\underline{u},\underline{\eta})} = \{P_1 + mP_0 + \underline{u}Q_0 + \underline{\eta}Q_1\}$	$(1+m^2(\phi_\xi)^2)(1-m^2)\phi_{\xi\xi} = 4\phi_\xi(\underline{\eta}\underline{u} - m\underline{\eta}\Lambda_{\xi\xi})$, $(1-m^2)\Lambda_{\xi\xi} = -(m\underline{u}+\underline{\eta})(\phi_\xi)^2 - m\underline{u} + \underline{\eta}$
$\mathcal{L}_{(6,m;\underline{u},\underline{\eta})} = \{Z + mP_0 + \underline{u}Q_0 + \underline{\eta}Q_1\}$	$(m^2-1)\left(1+\frac{1}{m^2}(F_x)^2\right)F_{xx} = 4m\underline{\eta}\Lambda_{xx}$, $(m^2-1)\Lambda_{xx} = m\underline{u}\left(1+\frac{1}{m^2}(F_x)^2\right) - \frac{2}{m}\underline{\eta}F_x$
$\mathcal{L}_{(7,m;\underline{u},\underline{\eta})} = \{Z + mP_1 + \underline{u}Q_0 + \underline{\eta}Q_1\}$	$(m^2+1)^2F_{tt} = m^2\underline{\eta}\underline{u}F_t$, $(m^2+1)\Lambda_{tt} = m\underline{\eta} + \frac{2}{m}\underline{u}F_t - \frac{1}{m}\underline{\eta}(F_t)^2$
$\mathcal{L}_{(8,m,n;\underline{u},\underline{\eta})} = \{Z + mP_0 + nP_1 + \underline{u}Q_0 + \underline{\eta}Q_1\}$	$\left(1+\frac{1}{m^2}(F_\xi)^2\right)n^2F_{\xi\xi} = (m^2-1)F_{\xi\xi} - 4m\underline{\eta}\left(1-\frac{n}{m}F_\xi\right)\Lambda_{\xi\xi}$, $(1-m^2+n^2)\Lambda_{\xi\xi} = -\frac{1}{m^2}(m\underline{u}+n\underline{\eta})(F_\xi)^2 + \frac{2}{m}\underline{\eta}F_\xi + (n\underline{\eta} - m\underline{u})$

Subalgebras L_2 and L_3 lead to trivial linear solutions in x and t, respectively, for both the bosonic field ϕ and the fermionic field ψ.

For the subalgebra $L_{4,m}$, we identify two distinct cases. If $m = 1$ or $m = -1$, then the fields ϕ and ψ are each found to be arbitrary functions of the symmetry variable $\xi = t - mx$:

$$\phi = \phi(t - mx), \qquad \psi = \psi(t - mx), \qquad \text{where} \quad m = \pm 1. \tag{23}$$

This solution represents a travelling wave of arbitrary profile for both the bosonic field ϕ and the fermionic field ψ. In particular, if a field has compact support, then it constitutes a solitary wave. If $m \neq -1, 0, 1$, then the fields ϕ and ψ become linear functions of $t - mx$.

Similarly, for the subalgebra $L_{6,m}$, we consider the two distinct cases. If $m = 1$ or $m = -1$, then the function F and the field ψ are each found to be arbitrary functions of x. Thus, the solution is

$$\phi(x,t) = m(F(x) + t), \qquad \psi = \psi(x), \qquad \text{where} \quad m = \pm 1. \tag{24}$$

The nature of the solution thus depends upon the properties of the function F. If $m \neq -1, 0, 1$, then we obtain the travelling wave solution

$$\phi(x,t) = \frac{1}{m}(t + C_1 x + C_2), \qquad \psi(x) = \eta_1 x + \eta_2, \tag{25}$$

where C_1, C_2 are bosonic and η_1, η_2 are fermionic.

The subalgebra $L_{7,m}$ leads to a solution similar to that found in (25).

For the subalgebra $L_{8,m,n}$, we distinguish two cases. If $m^2 - n^2 = 1$, then the solution depends on two arbitrary functions of the symmetry variable $\xi = x - \frac{n}{m}t$:

$$\phi(x,t) = F\left(x - \frac{n}{m}t\right) + \frac{t}{m}, \qquad \psi = \psi\left(x - \frac{n}{m}t\right). \tag{26}$$

The character of the solution therefore depends on the nature of the functions F and ψ. Among the physically interesting possibilities are bumps, kinks and doubly periodic solutions. For instance, the functions $F(\xi) = \psi(\xi) = \arctan(\xi)$ correspond in fixed time to kink solutions for both fields. It should be noted that, since the energy of the function increases with time, it will tend to become infinite. This problem can be avoided for certain functions by setting an appropriate bound on the time. If $m^2 - n^2 \neq 1$, then the solution is

$$\phi(x,t) = C_1\left(x - \frac{n}{m}t\right) + C_2 + \frac{t}{m}, \qquad \psi(x,t) = \eta_1\left(x - \frac{n}{m}t\right) + \eta_2, \tag{27}$$

where C_1, C_2 are bosonic and η_1, η_2 are fermionic.

The subalgebra $\mathcal{L}_{(2;\underline{u},\underline{\eta})}$ leads to the following solution, which is quadratic for ψ:

$$\phi(x) = C_1 x + C_2, \qquad \psi(x,t) = \tfrac{1}{2}\left(1 + C_1^2\right)\underline{u}x^2 + \underline{\eta}xt + \tfrac{1}{2}\underline{u}t^2 + \eta_1 x + \eta_2, \tag{28}$$

where C_1, C_2 are bosonic and η_1, η_2 are fermionic.

Let us now consider the subalgebra $\mathcal{L}_{(3;\underline{u},\underline{\eta})}$. The reduced equation for ϕ, given by

$$\phi_{tt} = 4\phi_t\underline{\eta}\underline{u}, \tag{29}$$

can be integrated once to give the formula

$$\phi_t = K_0\,\mathrm{e}^{4\underline{\eta}\underline{u}t}, \tag{30}$$

where K_0 is a bosonic constant. In the case where $\underline{\eta}\underline{u} = 0$, we obtain the quadratic solution

$$\phi(t) = K_0 t + K_1, \qquad \psi(x,t) = \tfrac{1}{2}\left(1 - K_0^2\right)\underline{\eta}t^2 + \underline{u}xt + \tfrac{1}{2}\underline{\eta}x^2 + \eta_1 x + \eta_2, \tag{31}$$

where K_1, K_2 are bosonic and η_1, η_2 are fermionic. For the case $\underline{\eta}\underline{u} \neq 0$, the only possible solution for ϕ would be

$$\phi(t) = \frac{K_0}{4\underline{\eta}\underline{u}}\,\mathrm{e}^{4\underline{\eta}\underline{u}t} + K_1, \tag{32}$$

where K_1 is bosonic. However, the second reduced equation

$$\Lambda_{tt} = (1 - (\phi_t)^2)\underline{\eta} \tag{33}$$

would require us to integrate an exponential similar to that in equation (30) twice, which would lead to a factor of $(\underline{\eta}\underline{u})^2 = 0$ in the denominator of the expression for Λ. Therefore, such a solution does not exist in this case.

For the subalgebra $\mathcal{L}_{(4,m;\underline{u},\underline{\eta})}$, we consider a number of cases. If $m = 1$ or $m = -1$, then the reduced equations from table 3 read

$$(1)\quad 4\phi_\xi(\underline{\eta}\underline{u} - m\underline{\eta}\Lambda_{\xi\xi}) = 0 \qquad \text{and} \qquad (2)\quad (m\underline{u} + \underline{\eta})(\phi_\xi)^2 + m\underline{u} - \underline{\eta} = 0. \tag{34}$$

The first equation (1) imposes the requirement that either $\phi_\xi = 0$, $\underline{\eta} = 0$ or $\Lambda_{\xi\xi} = \underline{u}$. In the first instance, the field ϕ is simply a bosonic constant while Λ is an arbitrary function of ξ. The case where $\underline{\eta} = 0$ leads to the travelling wave solution $\phi(x,t) = \pm\mathrm{i}(t - mx) + K_0$, where again Λ is an arbitrary function of ξ. If $\Lambda_{\xi\xi} = \underline{u}$, we obtain the quadratic solution

$$\begin{aligned} \phi(x,t) &= K_1(t - mx) + K_0, \\ \psi(x,t) &= \frac{1}{2m}\underline{u}(t - mx)^2 - \frac{1}{2}m\underline{u}x^2 + \underline{u}xt + \frac{1}{2}\underline{\eta}x^2 + \eta_1(t - mx) + \eta_2, \end{aligned} \tag{35}$$

where K_0, K_1 are bosonic parameters which must obey the relation $K_1^2(\underline{\eta} + m\underline{u}) = \underline{\eta} - m\underline{u}$ and η_1, η_2 are fermionic. For the case where $m \neq -1, 0, 1$, we obtain a situation similar to that for subalgebra $\mathcal{L}_{(3;\underline{u},\underline{\eta})}$ above. Solving the second reduced equation for $\Lambda_{\xi\xi}$ and substituting this factor into the first reduced equation, we obtain the following equation for ϕ_ξ:

$$(1 - m^2)^2\phi_{\xi\xi} = 4\phi_\xi\underline{\eta}\underline{u}, \tag{36}$$

which is solved by an exponential involving the factor $\underline{\eta}\underline{u}$. Once again, since we cannot integrate twice, a solution does not exist.

Let us consider the subalgebra $\mathcal{L}_{(6,m;\underline{u},\underline{\eta})}$. For the case where $m = \pm 1$, we obtain the solution

$$\phi(x) = \frac{1}{m}(t + C_1 x + C_0), \qquad \psi(x) = \frac{1}{2m}\underline{u}t^2 + \frac{1}{m}\underline{\eta}xt + \eta_1 x + \eta_2, \tag{37}$$

where C_0, C_1 are bosonic parameters which must obey the relation $\underline{u}\big(1 + C_1^2\big) - 2\underline{\eta}C_1 = 0$ and η_1, η_2 are fermionic. The case $m \neq -1, 0, 1$ leads us to the quadratic solution

$$\begin{aligned} \phi(x) &= \frac{1}{m}\left(t + \frac{2m^2\underline{\eta}\underline{u}}{m^2 - 1}x^2 + K_1 x + K_0\right), \\ \psi(x) &= \frac{1}{2m(m^2 - 1)}\big(m^2\underline{u} + K_1^2\underline{u} - 2K_1\underline{\eta}\big)x^2 + \frac{1}{m}\underline{\eta}xt + \frac{1}{2m}\underline{u}t^2 + \eta_1 x + \eta_2, \end{aligned} \tag{38}$$

where K_0, K_1 are bosonic constant parameters and η_1, η_2 are fermionic constant parameters.

For subalgebras $\mathcal{L}_{(7,m;\underline{u},\underline{\eta})}$ and $\mathcal{L}_{(8,m,n;\underline{u},\underline{\eta})}$, the reduced equations lead once again to exponentials involving $\underline{\eta}\underline{u}$ which must be integrated twice. Therefore, we do not obtain solutions for these cases.

5. Summary and concluding remarks

The main purpose of this paper has been to construct a supersymmetric generalization of the scalar Born–Infeld equation using a superspace formalism which involves one independent fermionic variable θ in addition to the bosonic spacetime coordinates in (1 + 1) dimensions. The extension was formulated in terms of a superfield Φ composed of two component fields: the original bosonic field ϕ and an additional fermionic field ψ. In its most general form, the extension constitutes a four-parameter family of supersymmetric equations involving the superfield Φ and its covariant and time derivatives. When split into its bosonic and fermionic components, this covariant equation is in turn equivalent to a system of two partial differential equations each involving the four parameters (a, b, c, d). For the case where $a = 1, b = 1, c = 0$ and $d = 1$, the symmetry Lie superalgebra L_S was found to be larger than for the general case. The one-dimensional subalgebras of L_S were classified systematically into conjugate classes, and a number of group-invariant solutions for this case of the SSBI equation have been obtained. These solutions include

(1) solutions which are polynomial functions of x and t for both ϕ and ψ;
(2) algebraic solutions expressed in terms of radicals;
(3) solutions expressed in terms of arbitrary functions of the symmetry variable.

In particular, a number of physically interesting solutions were found, including bumps, kinks and doubly periodic solutions. The polynomial solutions have infinite energy when they are considered to be functions on the entire space. However, they could be rendered physically meaningful by the imposition of boundary conditions.

The analysis described in this paper could be extended in several directions. First, we could extend our classification of the symmetry Lie superalgebra L_S so as to include two-dimensional subalgebras of L_S, which would allow us to seek for partially invariant solutions. This approach has been very successful in the search for more diverse classes of solutions [21, 22]. Second, we may consider using the method of conditional symmetries, which would include a search for multiple wave solutions expressed in terms of Riemann invariants. We could also attempt to generalize our analysis to that of solutions involving weak transversality

[24]. Finally, it may be noted that so far we have only explored in detail a specific case of the SSBI equation. Different values for the parameters a, b, c and d in the general form of the equations could be considered. These areas will constitute the subjects of future works.

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